THE THIN PLATE UNDER RANDOM LOADING

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The problem considered here has immediate application to the calculation of modern telescopic reflectors. For large telescopes, the bending of the reflecting surface of the mirror under its own weight leads to inadmissible distortion of the image. Special loading systems are used to reduce these bending deflections. The mirror is attached to the telescope tube at three points by hinged supports and on the back surface there is applied a system of unloading forces which act to reduce the bending of the mirror under its own weight. This suspension system permits the use of mirrors of admissible weight but with bending of the mirror surface not in excess of acceptable values. The mirror, from a mechanical point of view, is a continuous or a ribbed plate. For large mirrors, small errors in the unloading forces may lead to an inadmissible image surface distortion.

The solution given here may be applied to determine the normal deflections of a mirror (considered as a plate) under the action of incorrect unloading forces.

1. We denote by $\alpha(x, \xi)$ the function for determination of the support conditions of the plate. This function denotes the normal deflection of the plate at a point x, a two-dimensional vector coordinate, due to application of unit concentrated force perpendicular to the plate at the point ξ. The bending deflection of the plate under the action of a system of concentrated forces p_j at points ξ_j has the form

$$w = w(x, p_j) = \sum_j \alpha(x, \xi_j) p_j \tag{1.1}$$

We assume that the values of the forces p_j may have any value in the range $(-\sigma_j, +\sigma_j)$, where σ_j is some positive value. It is required to determine the maximum absolute deflection of the plate with the most unfavorable values of p_j. Denote this deflection by U. For its determination we have

$$U = \max |w(x, p_j)| \qquad (x \in s, \ |p_j| \leqslant \sigma_j) \tag{1.2}$$

Here s is the set of all points in the middle surface of the plate.

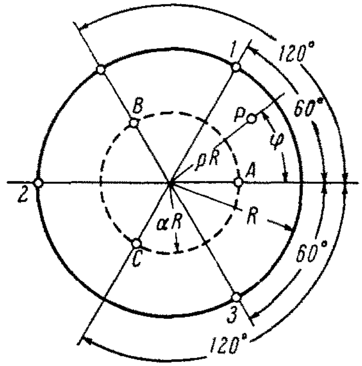

Fig. 1.

Since the bending of the plate, represented by the function $w(x, p_j)$, is continuous, an investigation of the maximum $|w|$ must lead to two cases: First the maximum $|w|$ is found for all variations in p_j, and then for variations in x

$$U = \max_x \max_p |w(x, p_j)| \qquad (x \in s, \ |p_j| \leqslant \sigma_j) \tag{1.3}$$

Then internal maximum is easily determined if the deflection is represented by (1.1):

$$\max_p \left| \sum_j \alpha(x, \xi_j) p_j \right| = \sum_j |\alpha(x, \xi_j)| \sigma_j \qquad (p_j \leqslant \sigma_j) \tag{1.4}$$

Actually, the modulus of the sum achieves its largest value when all terms take their largest values and have the same sign. Thus

$$U = \max_x \sum_j |\alpha(x, \xi_j)| \sigma_j \qquad (x \in s) \tag{1.5}$$

2. In the case of a large number of forces there is small probability of their being distributed in the most unfavorable manner. A more natural assumption is that the forces p_j are independent and have random values.

We determine the statistical characteristics of the plate deflection. We assume in addition to the above that the forces p_j are distributed according to a normal law with a mathematical expectation of zero and dispersion D_j. Since the plate deflection is a homogeneous linear function of the forces p_j, it is also distributed according to a normal law, with mathematical expectation of zero. The mean square value of deflection $\Delta(x)$ is given by the following formula [1]:

$$\Delta(x) = \left(\sum_j \alpha^2(x, \xi_j) D_j \right)^{1/2} \tag{2.1}$$

As a qualitative estimate of the error in deflection, one may use the maximum value of the doubled mean square

$$K = 2 \max \Delta(x) \qquad (x \in s) \tag{2.2}$$

since for a normal distribution random values of $w(x)$ with a probability of 0.95 are included in the interval $[-2\Delta(x), +2\Delta(x)]$.

3. We construct the influence function for a plate of constant stiffness D, hinge supported at three points A, B and C as shown in Fig. 1. The edge of the plate is assumed to be free of forces and moments, and the deflection of the plate at points A, B and C is equal to zero. The supports are at a distance aR from the center, where R is the plate radius and a is a ratio.

For determination of the influence function one must find the plate deflection at a point with coordinates rR, θ under the action of unit concentrated force at the point P (Fig. 1) with coordinates ρR, φ, and the support reactions A, B and C which in view of equilibrium conditions have the following values:

$$A = -\frac{1}{3}\left[1 + 2\frac{\rho}{a}\cos\varphi\right], \quad B = -\frac{1}{3}\left[1 + 2\frac{\rho}{a}\cos\left(\varphi - \frac{2\pi}{3}\right)\right] \tag{3.1}$$
$$C = -\frac{1}{3}\left[1 + 2\frac{\rho}{a}\cos\left(\varphi + \frac{2\pi}{3}\right)\right]$$

The problem of plate deflections under any system of arbitrarily distributed concentrated forces was considered by Lur'e [2], by Bassali [3], and others, and the solution for the problem here discussed may be obtained as a special case of the corresponding expressions in [1] and [2]. Nevertheless, it is expedient to obtain the solution by means of some evident simplifications arising from symmetry in the location of supports A, B and C. The expression for the influence function will be sought in the form

$$\alpha(r, \theta; \rho, \varphi) = \frac{R^2}{8\pi D} w = \frac{R^2}{8\pi D}(w_0 + w^*) \tag{3.2}$$

Here $R^2 w_0 / 8\pi D$ is a particular solution of the equation for plate bending, taking into account the action of the concentrated forces

$$w_0 = z^2 \ln z + A z_1^2 \ln z_1 + B z_2^2 \ln z_2 + C z_3^2 \ln z_3 \tag{3.3}$$

$$z = \sqrt{r^2 + \rho^2 - 2\rho r \cos(\theta - \varphi)}, \qquad z_2 = \sqrt{r^2 + a^2 - 2ar\cos(\theta - 2/3\pi)} \tag{3.4}$$
$$z_1 = \sqrt{r^2 + a^2 - 2ar\cos\theta}, \qquad z_3 = \sqrt{r^2 + a^2 - 2ar\cos(\theta + 2/3\pi)}$$

and w^* is a biharmonic function with no singularities in derivatives with respect to either r or θ, up to the third order in the entire region

$r < 1$.

We take an expression for w^* in the form of a series [4]

$$w^* = R_0 + \sum_{n=1}^{\infty} (R_n \cos n\theta + R_n' \sin n\theta) \tag{3.5}$$

where

$$R_n = (A_n + B_n r^2) r^n \quad (n = 0, 1, 2 \ldots), \qquad R_n' = (A_n' + B_n' r^2) r^n \quad (n = 1, 2, \ldots)$$

By making use of the known expansion

$$\ln z = \ln r - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\rho}{r}\right)^n \cos n(\theta - \varphi) \qquad (r > \rho) \tag{3.6}$$

as well as the expression for support reactions according to (3.1), we obtain an expression for w_0 in Fourier series in the angle θ, which holds for $r > \rho$, a:

$$\begin{aligned} w_0 = {} & (\rho^2 - a^2)(\ln r + 1) - \frac{\rho}{2r}(\rho^2 - a^2)(\cos\theta\cos\varphi + \sin\theta\sin\varphi) + \\ & + \sum_{n=2}^{\infty} \frac{1}{n} \left\{ \cos n\theta \left[\left(\frac{\rho}{r}\right)^n \left(\frac{r^2}{n-1} - \frac{\rho^2}{n+1}\right) \cos n\varphi - \left(\frac{a}{r}\right)^n \left(\frac{r^2}{n-1} - \frac{a^2}{n+1}\right) C_n \right] + \right. \\ & \left. + \sin n\theta \left[\left(\frac{\rho}{r}\right)^n \left(\frac{r^2}{n-1} - \frac{\rho}{n+1}\right) \sin n\varphi - \left(\frac{a}{r}\right)^n \left(\frac{r^2}{n-1} - \frac{a^2}{n+1}\right) S_n \right] \right\} \end{aligned} \tag{3.7}$$

The notation

$$C_n = \frac{2}{3} \frac{\rho}{a} \cos\varphi \left(1 - \cos\frac{2\pi n}{3}\right) + \frac{1}{3}\left(1 + 2\cos\frac{2\pi n}{3}\right)$$
$$S_n = \frac{2}{3}\sqrt{3} \sin\varphi \sin\frac{2\pi n}{3}$$

has been introduced here.

Since the edge of the plate is free of forces and bending moments, it is necessary to satisfy the following conditions [4]:

$$\begin{aligned} & \Delta\alpha - (1 - \nu)\left(\frac{1}{r}\frac{\partial\alpha}{\partial r} + \frac{1}{r^2}\frac{\partial^2\alpha}{\partial\theta^2}\right) = 0 \quad \left(\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) \\ & \frac{\partial}{\partial r}\Delta\alpha + (1 - \nu)\left(\frac{1}{r^2}\frac{\partial^3\alpha}{\partial r\partial\theta^2} - \frac{1}{r^3}\frac{\partial^2\alpha}{\partial\theta^2}\right) = 0 \quad \text{when } r = 1 \end{aligned} \tag{3.8}$$

Substitution of Expression (3.2) for the influence function into (3.8), taking account of the expansions referred to, leads to a system of equations for the values of A_n and B_n. The solution of this system is

$$B_0 = \frac{1-\nu}{2(1+\nu)}(\rho^2 - a^2)$$

$$B_1 = \frac{1-\nu}{2(3-\nu)}\rho(\rho^2 - a^2)\cos\varphi, \qquad B_1' = \frac{1-\nu}{2(3+\nu)}\rho(\rho^2 - a^2)\sin\varphi \quad \text{when } n \geqslant 2$$

$$B_n = -\frac{1-\nu}{3+\nu}\left[\rho^n\left(\frac{1}{n} - \frac{\rho^2}{n+1}\right)\cos n\varphi - a^n\left(\frac{1}{n} - \frac{a^2}{n+1}\right)C_n\right]$$

$$A_n = -\frac{n+1}{n}B_n + \frac{3+\nu}{1-\nu}\frac{1}{n^2(n-1)}(\rho^n\cos n\varphi - a^n C_n)$$

$$B_n' = -\frac{1-\nu}{3+\nu}\left[\rho^n\left(\frac{1}{n} - \frac{\rho^2}{n+1}\right)\sin n\varphi - a^n\left(\frac{1}{n} - \frac{a^2}{n+1}\right)S_n\right]$$

$$A_n' = -\frac{n+1}{n}B_n' + \frac{3+\nu}{1-\nu}\frac{1}{n^2(n-1)}(\rho^n\sin n\varphi - a^n S_n)$$

To find the constants A_0, A_1, and A_1', we write w^* in the form

$$w^* = A_0 + (A_1\cos\theta + A_1'\sin\theta)\,r + v \tag{3.9}$$

where we have

$$v = \sum_{n=2}^{\infty}(A_n\cos n\theta + A_n'\sin n\theta)\,r^n + \sum_{n=0}^{\infty}(B_n\cos n\theta + B_n'\sin n\theta)\,r^{n+2} \tag{3.10}$$

corresponding to Formula (3.5).

We substitute (3.9) in (3.2) and satisfy the support conditions at the points A, B and C:

$$\begin{gathered}A_0 + A_1 a + w_0(A) + v(A) = 0\\ A_0 + A_1 a\cos\frac{2\pi}{3} + A_1'\sin\frac{2\pi}{3} + w_0(B) + v(B) = 0\\ A_0 + A_1 a\cos\frac{2\pi}{3} - A_1'\sin\frac{2\pi}{3} + w_0(C) + v(C) = 0\end{gathered} \tag{3.11}$$

Here $w_0(A)$, ..., $v(C)$ are values of the functions w_0 and v at the points $A(r = a,\ \theta = 0)$, $B(r = a,\ \theta = 2\pi/3)$, and $C(r = a,\ \theta = -2\pi/3)$.

The solution of the system (3.11) is

$$\begin{gathered}A_0 = -\frac{1}{3}[w_0(A) + w_0(B) + w_0(C)] - \frac{1}{3}[v(A) + v(B) + v(C)]\\ A_1' = -\frac{1}{a\sqrt{3}}[w_0(B) - w_0(C) + v(B) - v(C)]\\ A_1 = -\frac{1}{a}[A_0 + w_0(A) + v(A)]\end{gathered} \tag{3.12}$$

Thus, all values of the quantities A_n, B_n, A_n', B_n' entering into the expression for w^* have been found; w^* is evidently a function of the coordinates of the point of application (ρ, φ) of the unit concentrated force. Therefore

$$w = w(r, \theta; \rho, \varphi)$$

4. Equation (1.5) simplifies when $\sigma_j = \sigma_0$ = const. In addition, if the forces p_j are applied to the plate at points of a regular network in large number, the sum may be approximated by an integral

$$U \approx \sigma_0 n \max \frac{1}{S}\int_S |\alpha(x, \xi)|\, d\xi \qquad (x \in S) \tag{4.1}$$

where S is the plate area, n the full number of concentrated forces including the support reactions, and $d\xi$ is the differential of area. Substitution of the integral for the sum facilitates the calculation to a certain extent, since the integral does not take into account the actual system of forces applied to the plate (at the corners of a square, triangle, or other network). It is evident that the error introduced by this substitution is small, when n is large.

By substitution of Expression (3.2) into (4.1) with $d\xi = R^2\rho d\rho d\varphi$, we obtain

$$U = \frac{R^2\sigma_0 n}{8\pi D} \max\left(\frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi} |w(r, \theta; \rho, \varphi)|\, \rho d\varphi d\rho\right) \qquad \begin{pmatrix} r \leqslant 1 \\ |\theta| \leqslant \pi \end{pmatrix} \tag{4.2}$$

Numerical calculation shows that the quantity in curved brackets achieves its maximum value at points 1, 2 and 3. Therefore

$$U = \frac{R^2\sigma_0 n}{8\pi D}\psi \qquad \left(\psi = \frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi} |w(1, \pi; \rho, \varphi)|\, \rho d\varphi d\rho\right) \tag{4.3}$$

Since the influence function depends on a, ψ is a function of a. The relation is shown in Fig. 2.

Analogous considerations lead to the expressions

$$K = \frac{R^2 \sqrt{D_0 n}}{4\pi D}\chi, \qquad \chi = \left(\frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi} [w(1, \pi; \rho, \varphi)]^2 \rho d\varphi d\rho\right)^{1/2} \tag{4.4}$$

$$(D_0 = D_j = \text{const})$$

The relation of χ to a is shown also in Fig. 2.

For application to the problem of unloading a telescope mirror one must set

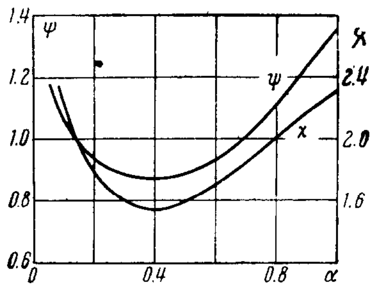

Fig. 2.

$$\sigma_0 = \frac{P}{n}\delta, \qquad D_0 = \left(\frac{P}{n}\varepsilon\right)^2 \tag{4.5}$$

where P is the weight of the mirror, δ and ε are maximum and mean square errors in the unloading forces. Substitution of (4.5) into Formulas (4.3) and (4.4) gives the following results:

$$U = \frac{R^2 P}{8\pi D}\delta\psi, \qquad K = \frac{R^2 P}{4\pi D\sqrt{n}}\varepsilon\chi \tag{4.6}$$

Using the Formulas (4.6), calculations were made for three mirrors (R = 1.35, 3.06, 2.55), supposed to be continuous plates of constant thickness, with the material constants: density 2.25 g/cm^3, Young's modulus 0.7 x 10^6 kg/cm^2, Poisson's ratio 0.25. In addition, it was supposed that $\varepsilon = 0.25 \times 10^{-2}$, $\delta = 0.5 \times 10^{-2}$, $V = 5 \times 10^{-6}$ cm.

Results of the calculation for U and K are as follows:

R, м	Ra, м	a	h, м	n	U/V	K/V
1.35	0.78	0.58	0.336	27	1.29	0.46
3.05	1.83	0.60	1	50	3.87	0.95
2.55	1.53	0.60	0.65	36	4.45	1.28

If the allowable mirror deflection is V, then even for such small errors in the unloading forces as were taken into account, the mirror deflections have values of the same order as those assumed.

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